LIBERTY PAPER SET

STD. 12 : Mathematics

Full Solution

Time : 3 Hours **ASSIGNTMENT PAPER 15**

PART A

1. (B) **2.** (B) **3.** (D) **4.** (C) **5.** (D) **6.** (C) **7.** (B) **8.** (C) **9.** (B) **10.** (A) **11.** (B) **12.** (D) **13.** (A) **14.** (D) **15.** (C)
16. (D) **17.** (C) **18.** (A) **19.** (D) **20.** (A) **21.** (D) **22.** (D) **23.** (A) **24.** (D) **25.** (A) **26.** (C) **27.** (B)
28. (C) **29.** (B) **30.** (C) **31.** (C) **32.** (B) **33.** (D) **34.** (D) **35.** (C) **36.** (C) **37.** (D) **38.** (C) **39.** (D)
40. (C) **41.** (C) **42.** (A) **43.** (B) **44.** (B) **45.** (B) **46.** (C) **47.** (B) **48.** (D) **49.** (C) **50.** (B)

PART B

SECTION A

1.

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$= \tan^{-1}\left(\sqrt{\tan^2\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\left|\tan\frac{x}{2}\right|\right)$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right)$$

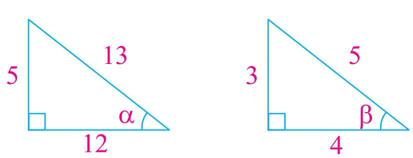
$$\left(\because\ 0 < x < \pi \Rightarrow 0 < \frac{x}{2} < \frac{\pi}{2}\ \therefore\ \tan\frac{x}{2} > 0\right)$$

$$= \frac{x}{2}\quad\left(\because\ \frac{x}{2} \in \left(0, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

2.

$$\text{L.H.S.} = \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5}$$

$$\cos^{-1}\frac{12}{13} = \alpha \quad , \quad \sin^{-1}\frac{3}{5} = \beta$$



$$\therefore \cos\alpha = \frac{12}{13}, \sin\alpha = \frac{5}{12} \quad \Big| \quad \sin\beta = \frac{3}{5}, \cos\beta = \frac{4}{5}$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \left(\frac{5}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right)$$

$$= \frac{20}{65} + \frac{36}{65}$$

$$= \frac{56}{65}$$

$$\therefore \quad \alpha + \beta = \sin^{-1}\frac{56}{65}$$

$$\therefore \quad \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$$

3.

$$y = \cos x^3 \cdot \sin^2(x^5)$$

$$\frac{dy}{dx} = \cos x^3 \cdot \frac{d}{dx}\sin^2(x^5) + \sin^2(x^5)\frac{d}{dx}\cos x^3$$

$$= \cos x^3 \cdot 2\sin(x^5) \cdot \frac{d}{dx}(\sin(x^5))$$

$$+ \sin^2(x^5)(-\sin x^3)\frac{d}{dx}(x^3)$$

$$= \cos x^3 \cdot 2\sin(x^5) \cdot \cos x^5 \cdot 5x^4 - \sin^2(x^5)\sin x^3 \cdot 3x^2$$

$$= 5x^4 \cdot \cos x^3 \cdot 2\sin(x^5) \cdot \cos x^5$$

$$- 3x^2 \cdot \sin^2(x^5) \cdot \sin x^3$$

$$= 5x^4 \cdot \cos x^3 \sin(2x^5) - 3x^2 \cdot \sin^2(x^5)\sin x^3$$

4.

Method 1 :

$$I = \int \frac{\cos x}{1+\cos x}\, dx$$

$$= \int \frac{\cos x}{1+\cos x} \times \frac{(1-\cos x)}{(1-\cos x)}\, dx$$

$$I = \int \frac{\cos x - \cos^2 x}{\sin^2 x}\, dx$$

$$= \int \frac{\cos x}{\sin^2 x}\, dx - \int \frac{\cos^2 x}{\sin^2 x}\, dx$$

$$= \int \cot x \cdot \operatorname{cosec} x\, dx - \int \cot^2 x\, dx$$

$$= \int \cot x \cdot \operatorname{cosec} x\, dx - \int (\operatorname{cosec}^2 x - 1)dx$$

$$= \int \cot x \cdot \operatorname{cosec} x\, dx - \int \operatorname{cosec}^2 x\, dx + \int 1\, dx$$

$$I = -\operatorname{cosec} x + \cot x + x + c$$

$$I = x - (\operatorname{cosec} x - \cot x) + c$$

$$I = x - \left(\frac{1 - \cos x}{\sin x}\right) + c$$

$$= x - \left(\frac{2\sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}}\right) + c$$

$$= x - \tan \frac{x}{2} + c$$

Method 2 :

$$\int \frac{\cos x}{1+\cos x}\, dx$$

$$= \int \frac{1+\cos x - 1}{1+\cos x}\, dx$$

$$= \int \frac{1+\cos x}{1+\cos x}\, dx - \int \frac{1}{1+\cos x}\, dx$$

$$= \int 1\, dx - \int \frac{1}{2\cos^2 \frac{x}{2}}\, dx$$

$$= \int 1\, dx - \frac{1}{2}\int \sec^2 \frac{x}{2}\, dx$$

$$= x - \frac{1}{2}\, \frac{\tan \frac{x}{2}}{\frac{1}{2}} + c$$

$$= x - \tan \frac{x}{2} + c$$

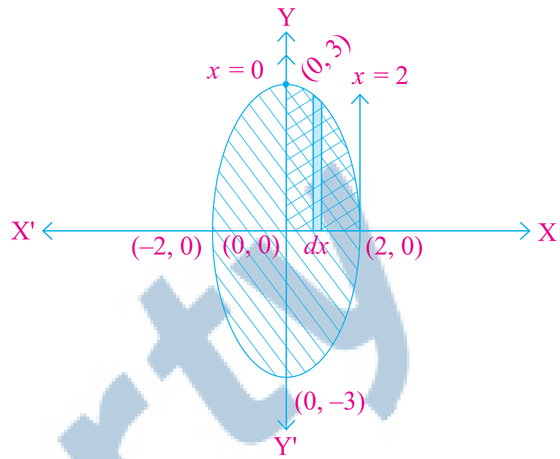
5.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$a^2 = 4,\ a = 2$

$b^2 = 9,\ b = 3$

$b > a$



Required Area :

A = 4 × Area bounded on the first quadrant

$\therefore$ A = 4|I|

Now, $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$$\therefore y^2 = 9\left[1 - \frac{x^2}{4}\right]$$

$$= \frac{9}{4}(4 - x^2)$$

$$\therefore y = \frac{3}{2}\sqrt{4 - x^2}$$

$$I = \int_0^2 y\, dx$$

$$I = \int_0^2 \frac{3}{2}\sqrt{4-x^2}\, dx$$

$$I = \frac{3}{2}\int_0^2 \sqrt{4-x^2}\, dx$$

$$= \frac{3}{2}\int_0^2 \sqrt{2^2-x^2}\, dx$$

$$I = \frac{3}{2}\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$I = \frac{3}{2}\left[\left(\frac{2}{2}(0) + 2\sin^{-1}(1)\right) - (0)\right]$$

$$I = \frac{3}{2} \cdot 2 \cdot \frac{\pi}{2}$$

$$I = \frac{3\pi}{2}$$

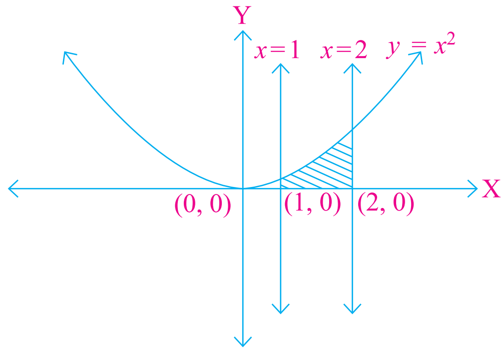
Now, A = 4|I|

$$= 4\left|\frac{3\pi}{2}\right|$$

$\therefore$ A = 6π sq. units

6.

$x^2 = y$



Area, A = |I|

$\therefore\ I = \int_1^2 y\,dx$

$\therefore\ I = \int_1^2 x^2\,dx$

$\therefore\ I = \left[\frac{x^3}{3}\right]_1^2$

$\therefore\ I = \frac{1}{3}\left((2)^3 - (1)^3\right)$

$\therefore\ I = \frac{7}{3}$

Now, $A = |I| = \left|\frac{7}{3}\right|$

$\therefore\ A = \frac{7}{3}$ sq. units

7.

$y \log y\,dx - x\,dy = 0$

$\therefore\ x\,dy = y \log y\,dx$

$\therefore\ \frac{dy}{y \log y} = \frac{dx}{x}$

→ Integrate both sides,

$\therefore\ \int \frac{dy}{y \log y} = \int \frac{dx}{x}$

$\therefore\ \int \frac{\frac{1}{y}\,dy}{\log y} = \int \frac{dx}{x}$

$\therefore\ \int \frac{\frac{d}{dy}(\log y)}{\log y}\,dy = \int \frac{dx}{x}$

$\therefore\ \log|\log(y)| = \log|x| + \log|c|$

$\therefore\ \log|\log y| = \log|x \cdot c|$

$\therefore\ \log y = x \cdot c$

$\therefore\ y = e^{xc};$

Which is uniques solution of given differentiation equation.

8.

$\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$

$\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$

$2\vec{a} - \vec{b} + 3\vec{c}$

$= 2\hat{i} + 2\hat{j} + 2\hat{k} - 2\hat{i} + \hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k}$

$= 3\hat{i} - 3\hat{j} + 2\hat{k}$

$2\vec{a} - \vec{b} + 3\vec{c}$ parallel vector,

$= \frac{2\vec{a} - \vec{b} + 2\vec{c}}{|2\vec{a} - \vec{b} + 3\vec{c}|}$

$= \frac{3\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{9+9+4}}$

$= \frac{3}{\sqrt{22}}\hat{i} - \frac{3}{\sqrt{22}}\hat{j} + \frac{2}{\sqrt{22}}\hat{k}$

9.

Here, $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$

$\therefore\ L : \vec{r} = 2\hat{i} + \hat{j} - 3\hat{k} + \lambda(2\hat{i} + 5\hat{j} - 3\hat{k})$

$\vec{b_1} = 2\hat{i} + 5\hat{j} - 3\hat{k}$

and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

$\therefore\ M : \vec{r} = (-2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu(-\hat{i} + 8\hat{j} + 4\hat{k})$

$\vec{b_2} = -\hat{i} + 8\hat{j} + 4\hat{k}$

$\vec{b_1} \cdot \vec{b_2} = (2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (-\hat{i} + 8\hat{j} + 4\hat{k})$

$= -2 + 40 - 12$

$= 26$

$|\vec{b_1}| = \sqrt{4+9+25}$

$= \sqrt{38}$

$|\vec{b_2}| = \sqrt{1+64+16}$

$= \sqrt{81}$

$= 9$

If the angle between L and M,

$\cos\alpha = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}$

$\cos\alpha = \frac{|26|}{\sqrt{38}\ 9}$

$\therefore\ \alpha = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$

Therefore, the angle between two lines is $\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$.

10.

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$$

$\therefore$ L : $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k})$
$+ \lambda(-3\hat{i} + 2k\hat{j} + 2\hat{k}),\ \lambda \in R$

$\vec{b_1} = -3\hat{i} + 2k\hat{j} + 2\hat{k}$

But, $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$

$\therefore$ M : $\vec{r} = (\hat{i} + \hat{j} + 6\hat{k})$
$+ \mu(3k\hat{i} + \hat{j} - 5\hat{k}),\ \mu \in R$

$\vec{b_2} = 3k\hat{i} + \hat{j} - 5\hat{k}$

Both lines are perpendicular to each other.

$\therefore\ \vec{b_1} \cdot \vec{b_2} = 0$

$\therefore\ (-3\hat{i} + 2k\hat{j} + 2\hat{k}) \cdot (3k\hat{i} + \hat{j} - 5\hat{k}) = 0$

$\therefore\ -9k + 2k - 10 = 0$

$\therefore\ -7k = 10$

$\therefore\ k = \frac{-10}{7}$

11.

Mother, father and son line up at random for a family picture

Suppose Mother $\to$ M
Father $\to$ F
Son $\to$ S

$\therefore$ Number of arrangrment of three persons in a row $= {}_3P_3$

$\therefore$ Possible outcomes $= {}_3P_3 = 3! = 6$

$\therefore$ S = {(M, F, S), (M, S, F), (F, M, S), (F, S, M), (S, M, F), (S, F, M)}

Event E : son on one end.

E = {(M, F, S), (F, M, S), (S, M, F), (S, F, M)}

$\therefore\ r = 4$

$\therefore\ P(E) = \frac{r}{n} = \frac{4}{6} = \frac{2}{3}$

Event F : father in middle

F : {(M, F, S), (S, F, M)}

$\therefore\ r = 2$

$\therefore\ P(F) = \frac{2}{6} = \frac{1}{3}$

$E \cap F$ = {(M, F, S), (S, F, M)}

$\therefore\ r = 2$

$\therefore\ P(E \cap F) = \frac{1}{3}$

$\therefore\ P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$

12.

Two dice are thrown $n = 36$

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Event A : Two numbers appearing on throwing two dice are different.

$\therefore\ r = 30$

$\therefore\ P(A) = \frac{30}{36} = \frac{5}{6}$

Event B : the sum of numbers on the dice is 4.

B = {(1, 3), (2, 2), (3, 1)}

$A \cap B$ = {(1, 3), (3, 1)}

$\therefore\ r = 2$

$\therefore\ P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

$\therefore\ P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{18}}{\frac{5}{6}} = \frac{1}{15}$

SECTION B

13.

Here, $A = R - \{3\}$, $B = R - \{1\}$, $f(x) = \left(\frac{x-2}{x-3}\right)$

$\forall\ x_1, x_2 \in A, \Rightarrow f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$

$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one function.

Suppose, $y \in B = R - \{1\}$

$y = f(x)$

$\therefore\ y = \frac{x-2}{x-3}$

$\therefore\ y(x-3) = x - 2$

$\therefore\ yx - 3y = x - 2$

$\therefore\ yx - x = 3y - 2$

$\therefore\ x(y-1) = 3y - 2$

$\therefore\ x = \frac{3y-2}{y-1} \in R - \{3\}$ (Domain)

Now, $f(x) = f\left(\frac{3y-2}{y-1}\right) = \frac{\frac{3y-2}{y-1}-2}{\frac{3y-2}{y-1}-3}$

$= \frac{3y-2-2y+2}{3y-2-3y+3} = y$

$\therefore$ For $\forall\ y \in B = R - \{1\}$

$x = \frac{3y-2}{y-1} \in A = R - \{3\}$ such that, $f(x) = y$.

$\therefore\ f$ is onto function.

Note : $f : R - \left\{-\frac{d}{c}\right\} \to R - \left\{\frac{a}{c}\right\}$;

$f(x) = \frac{ax+b}{cx+d}$ is always one-one and onto function.

14.

Total trust invests is ₹ 30,000.

Let a trust invests ₹ x in 5% interest per year and remaining fund ₹ $(30000 - x)$ in 7% interest per year.

(a) The trust fund must obtain an annual total interest ₹ 1,800

$5\% = \frac{5}{100}$

$7\% = \frac{7}{100}$

$\therefore\ [x\ \ 30000 - x]\begin{bmatrix}\frac{5}{100}\\ \frac{7}{100}\end{bmatrix} = [1800]$

$\therefore\ \left[\frac{5}{100}x + \frac{7}{100}(30000 - x)\right] = [1800]$

$\therefore\ \left[\frac{5x + 210000 - 7x}{100}\right] = [1800]$

$-2x + 210000 = 180000$

$\therefore\ 2x = 30000$

$\therefore\ x =$ ₹ 15000

Thus, to get annual interest ₹ 1,800,

first bond invests ₹ 15,000 and

Second bond invests $(30000 - 15000) =$ ₹ 15,000

(b) The trust fund must obtain an annual total interest ₹ 2,000.

$\therefore\ [x\ \ 30000 - x]\begin{bmatrix}\frac{5}{100}\\ \frac{7}{100}\end{bmatrix} = [2000]$

$\therefore\ \left[\frac{5x}{100} + \frac{7(30000 - x)}{100}\right] = [2000]$

$\therefore\ \frac{5x + 210000 - 7x}{100} = 2000$

$\therefore\ -2x + 210000 = 200000$

$\therefore\ -2x = -10000$

$\therefore\ x =$ ₹ 5,000.

Thus, to get annual interest ₹ 2,000,

first bond invests ₹ 5,000 and

Second bond invests $(30000 - 5000) =$ ₹ 25,000

15.

$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Co-factor of the element yz $A_{13} = (-1)^4 \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix}$

$= (1)(z - y) = (z - y)$

Co-factor of the element zx $A_{23} = (-1)^5 \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix}$

$= (-1)(z - x) = x - z$

Co-factor of the element xy $A_{33} = (-1)^6 \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix}$

$= (1)(y - x) = y - x$

$\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

$= (yz)(z - y) + (zx)(x - z) + (xy)(y - x)$

$= yz^2 - y^2z + zx^2 - z^2x + xy^2 - x^2y$

$= z(x^2 - y^2) + z^2(y - x) + xy(y - x)$

$= z[(x - y)(x + y)] + z^2(y - x) + xy(y - x)$

$= (y - x)(-z(x + y) + z^2 + xy)$

$= (y - x)(-zx - yz + z^2 + xy)$

$= (y - x)(z(z - x) - y(z - x))$

$= (y - x)(z - x)(z - y)$

$= (x - y)(y - z)(z - x)$

16.

Suppose, $u = (logx)^x$ and $v = x^{logx}$

$\therefore\ y = u + v$

Now, differentiate w.r.t. x,

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ .... (1)

Here, $u = (logx)^x$

Take both the sides log,

$log\ u = x\ log(logx)$

Now, differentiate w.r.t. x,

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}x$$

$$= x \times \frac{1}{x\log x} + \log(\log x)$$

$$\therefore \quad \frac{du}{dx} = u\left[\frac{1}{\log x} + \log(\log x)\right]$$

$$\frac{du}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] \quad \ldots\ldots (2)$$

Now, $v = x^{\log x}$

Take both the sides *log*

$\log v = \log x \cdot \log x$

Now, differentiate w.r.t. x,

$$\frac{1}{v}\frac{dv}{dx} = \log x\frac{d}{dx}\log x + \log x \cdot \frac{d}{dx}\log x$$

$$= \log x \times \frac{1}{x} + \log x \times \frac{1}{x}$$

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = \frac{2\log x}{x}$$

$$\therefore \quad \frac{dv}{dx} = v\left[\frac{2\log x}{x}\right]$$

$$\therefore \quad \frac{dv}{dx} = x^{\log x}\left[\frac{2\log x}{x}\right] \quad \ldots\ldots (3)$$

Put the value of equation (2) and (3) in equation (1),

$$\therefore \quad \frac{dy}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + x^{\log x}\left[\frac{2\log x}{x}\right]$$

$$= (\log x)^{x-1} + (\log x)^x \cdot \log(\log x) + x^{\log x - 1}[2\log x]$$

$$\frac{dy}{dx} = (\log x)^{x-1}[1 + \log x \cdot \log(\log x)] + 2x^{\log x - 1}(\log x)$$

17.

$$y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta, \ \theta \in \left[0, \frac{\pi}{2}\right]$$

$$\therefore \quad \frac{dy}{d\theta} = \frac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4(\cos^2\theta + \sin^2\theta) - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - 4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$\frac{dy}{d\theta} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

Here, $\theta \in \left[0, \frac{\pi}{2}\right] \Rightarrow \cos\theta \geq 0$

$\Rightarrow (4 - \cos\theta) > 0$

$\Rightarrow (2 + \cos\theta)^2 > 0$

$$\Rightarrow \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \geq 0$$

$$\Rightarrow \frac{dy}{d\theta} \geq 0$$

$\therefore$ f is increasing function of $\left[0, \frac{\pi}{2}\right]$.

18.

$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$

$\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$

The area of parellelogram,

$\Delta = |\vec{a} \times \vec{b}|$ ... (1)

$$\text{Now, } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$

$\vec{a} \times \vec{b} = 20\hat{i} + 5\hat{j} - 5\hat{k}$

$|\vec{a} \times \vec{b}| = \sqrt{400 + 25 + 25}$

$= \sqrt{450}$

$= 15\sqrt{2}$

From equation (1), $\Delta = 15\sqrt{2}$ sq. units.

19.

Two lines are parallel,

We have $\vec{a_1} = \hat{i} + 2\hat{j} - 4\hat{k}$,

$\vec{a_2} = 3\hat{i} + 3\hat{j} - 5\hat{k}$ and

$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

So, the distance between lines are

$$d = \left|\frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|}\right|$$

$$= \left|\frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}}{\sqrt{4+9+36}}\right|$$

$$= \frac{|-9\hat{i} + 14\hat{j} - 4\hat{k}|}{\sqrt{49}}$$

$$= \frac{\sqrt{293}}{\sqrt{49}}$$

$$= \frac{\sqrt{293}}{7} \text{ unit}$$

20.

$x + 2y \leq 120$
$x + y \geq 60$
$x - 2y \geq 0$
$x \geq 0$
$y \geq 0$

Objective function $Z = 5x + 10y$

$x + 2y = 120$ ... (i) $\quad x + y = 60$ ... (ii)

x	0	120	
y	60	0	

(0, 60) ×
(120, 0) ✓

x	0	60
y	60	0

(0, 60) ×
(60, 0) ✓

$x - 2y = 0$ ... (iii)

x	0	2
y	0	0

(0, 0) ×
(2, 1)

Solving equation (i) and (ii),

$$\begin{array}{r} x + 2y = 120 \\ x + y = 60 \\ - \quad - \quad - \\ \hline y = 60 \end{array} \quad (0, 60) \times$$

$\therefore x = 0$

Solving equation (ii) and (iii),

$$\begin{array}{r} x + y = 60 \\ x + 2y = 0 \\ - \quad + \quad - \\ \hline 3y = 60 \end{array} \quad (40, 20) ✓$$

$\therefore y = 20$

Put y = 20 in eq^n (iii)

$x - 2(20) = 0$

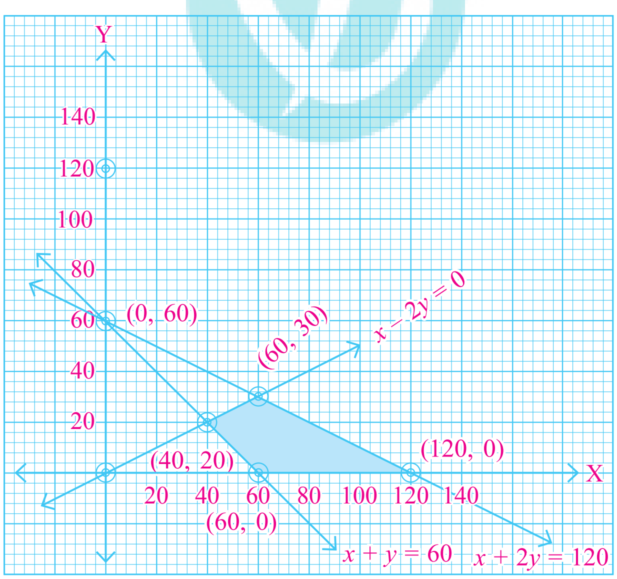
$\therefore x = 40$

Solving equation (i) and (iii),

$\therefore 4y = 120$

$\therefore y = 30$ $\qquad \therefore x = 60$

(60, 30) ✓ $\qquad$ (0, 0) ×



The shaded region in fig. is feasible region determined by the system of constrints which is bounded. The co-ordinates of corner points are (40, 20), (60, 30), (60, 0) and (120, 0).

Corner Point	Corresponding value of $Z = 5x + 10y$
(60, 30)	600 ← Maximum
(40, 20)	400
(60, 0)	300 ← Minimum
(120, 0)	600 ← Maximum

Thus, The maximum value of Z is 600 and Minimum value of Z is 300.

21.

60% students reside in a hostel

Event A : A student is hostelier

$$P(A) = \frac{60}{100}$$

40% are day scholars.

Event B : Student is not living in hostel.

$$P(B) = \frac{40}{100}$$

Event E : Student has an grade A

$$P(E) = P(A) \cdot P(E \mid A) + P(B) \cdot P(E \mid B)$$

$$= \frac{60}{100} \times \frac{30}{100} + \frac{40}{100} \times \frac{20}{100}$$

$$= \frac{18}{100} + \frac{8}{100}$$

$$= \frac{26}{100}$$

$$P(A \mid E) = \frac{P(A) \cdot P(E|A)}{P(E)}$$

$$= \frac{\frac{60}{100} \times \frac{30}{100}}{\frac{26}{100}}$$

$$= \frac{9}{13}$$

SECTION C

22.

Here, $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

$$A^T = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^T)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 3+3 & 3-2 & -1-4 \\ -2+3 & -2-2 & 1-5 \\ -4-1 & -5+1 & 2+2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}$$

$\therefore$ $P = P^T$

$\therefore$ P is symmetric matrix.

$$Q = \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 3-3 & 3+2 & -1+4 \\ -2-3 & -2+2 & 1+5 \\ -4+1 & -5-1 & 2-2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix}$$

$$Q^T = \begin{bmatrix} 0 & \frac{-5}{2} & \frac{-3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix}$$

$\therefore$ $Q = -Q^T$

$\therefore$ Q is skew symmetric matrix.

$$P + Q = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

$= A$

23.

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$$

$= 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$

$= 0 + 3(-2) + 5(1)$

$= -6 + 5$

$= -1 \neq 0$

$\therefore$ A^{-1} exists.

For finding *adj* A,

Co-factor of element 2 $A_{11} = (-1)^2\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = 1(-4+4) = 0$

Co-factor of element -3 $A_{12} = (-1)^3\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = (-1)(-6+4) = 2$

Co-factor of element 5 $A_{13} = (-1)^4\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1(3-2) = 1$

Co-factor of element 3 $A_{21} = (-1)^3\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = (-1)(6-5) = -1$

Co-factor of element 2 $A_{22} = (-1)^4\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = 1(-4-5) = -9$

Co-factor of element -4 $A_{23} = (-1)^5\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = (-1)(2+3) = -5$

Co-factor of element 1 $A_{31} = (-1)^4\begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 1(12-10) = 2$

Co-factor of element 1 $A_{32} = (-1)^5\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = (-1)(-8-15) = 23$

Co-factor of element -2 $A_{33} = (-1)^6\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 1(4+9) = 13$

$$\therefore\ adj\ A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

$$= \frac{1}{(-1)}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore\ A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Now, $2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

The equation can be represented as matrix form,

$$\therefore \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$\therefore$ AX = B

Where, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$\therefore \quad X = A^{-1}B$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution : $x = 1, y = 2, z = 3$

24.

$(x - a)^2 + (y - b)^2 = c^2$ .... (1)

Now, differentiate by x on both sides,

$$2(x - a) + 2(y - b)\frac{dy}{dx} = 0$$

$$\therefore (x - a) + (y - b)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-(x-a)}{y-b} \quad .... (2)$$

Now, differentiate by x on both sides,

$$\frac{d^2y}{dx^2} = -\left[\frac{(y-b)(1) - (x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b) + \frac{(x-a)(x-a)}{(y-b)}}{(y-b)^2}\right]$$

($\because$ From equation (2))

$$= -\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3}\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-c^2}{(y-b)^3} \quad (\because \text{ From equation (1)})$$

Now, $$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \left(\frac{x-a}{y-b}\right)^2\right]^{\frac{3}{2}}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{-c^2}{(y-b)^3}}$$

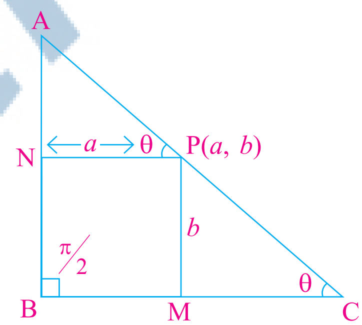
$$= \frac{-[c^2]^{\frac{3}{2}}}{\left[(y-b)^2\right]^{\frac{3}{2}}} \times \frac{(y-b)^3}{[c^2]}$$

$$= \frac{-c^3}{(y-b)^3} \times \frac{(y-b)^3}{c^2}$$

$$= -c, \quad c > 0$$

Which is constant independent of a and b.

25.



$\overline{AC}$ is hypotenuse of right angle triangle ΔABC.

Point $P(a, b)$ lies on the hypotenuse

Here, $\overline{PM} \perp \overline{BC}$ and $\overline{PN} \perp \overline{AB}$

Suppose, $\angle APN = \angle PCM = \theta$

ΔABC is right angle triangle $\theta \in \left(0, \frac{\pi}{2}\right)$

→ In ΔAPN,

$$\cos\theta = \frac{PN}{AP} = \frac{a}{AP}$$

$$\therefore \quad AP = \frac{a}{\cos\theta}$$

$$\therefore \quad AP = a\sec\theta$$

→ In ΔPMC,

$$\sin\theta = \frac{PM}{PC} = \frac{b}{PC}$$

$$\therefore \quad PC = \frac{b}{\sin\theta}$$

$$\therefore \quad PC = b\,\mathrm{cosec}\,\theta$$

Now, AC = AP + PC

$\therefore \quad AC = a\sec\theta + b\,\mathrm{cosec}\,\theta$ .... (1)

$f(\theta) = a \sec\theta + b \operatorname{cosec}\theta$

$\therefore \quad f'(\theta) = a \sec\theta \cdot \tan\theta - b \operatorname{cosec}\theta \cdot \cot\theta$

$\therefore \quad f''(\theta) = a(\sec\theta \cdot \sec^2\theta + \tan\theta \cdot \sec\theta \cdot \tan\theta)$

$\qquad - b(\operatorname{cosec}\theta\,(-\operatorname{cosec}^2\theta) + \cot\theta(-\operatorname{cosec}\theta \cot\theta))$

$\therefore \quad f''(\theta) = a(\sec^3\theta + \sec\theta \cdot \tan^2\theta) + b(\operatorname{cosec}^3\theta + \operatorname{cosec}\theta \cot^2\theta)$

$\therefore \quad f''(\theta) > 0 \qquad \left(\because 0 < \theta < \frac{\pi}{2}\right)$

→ For minimum length of hypotenuse,

$f'(\theta) = 0$

$\therefore \quad a \sec\theta \tan\theta - b \operatorname{cosec}\theta \cot\theta = 0$

$\therefore \quad a \sec\theta \tan\theta = b \operatorname{cosec}\theta \cot\theta$

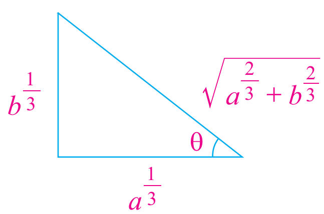
$\therefore \quad \frac{a}{\cos\theta}\,\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}\,\frac{\cos\theta}{\sin\theta}$

$\therefore \quad \frac{a}{\cos^3\theta} = \frac{b}{\sin^3\theta}$

$\therefore \quad \frac{\sin^3\theta}{\cos^3\theta} = \frac{b}{a}$

$\therefore \quad \tan^3\theta = \frac{b}{a}$

$\therefore \quad \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$



$\sec\theta = \frac{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}}$

$\operatorname{cosec}\theta = \frac{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}}$

Put this value in equation (1),

$\therefore \text{AC} = \frac{a\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{a^{\frac{1}{3}}} + \frac{b\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}}{b^{\frac{1}{3}}}$

$\therefore \text{AC} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}} \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{1}$

$\therefore \text{AC} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

$\therefore$ Minimum length of hypotenuse $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

26.

➪ $I = \int \frac{\sqrt{x^2+1}\,[\log(x^2+1) - 2\log x]}{x^4}\,dx$

$= \int \frac{\sqrt{x^2+1}\,[\log(x^2+1) - \log x^2]}{x^4}\,dx$

$= \int \frac{\sqrt{x^2+1}}{x^4} \log\left(\frac{x^2+1}{x^2}\right) dx$

$= \int \frac{x\sqrt{1+\frac{1}{x^2}}}{x^4} \log\left(1+\frac{1}{x^2}\right) dx$

$I = \int \frac{\sqrt{1+\frac{1}{x^2}}}{x^3} \log\left(1+\frac{1}{x^2}\right) dx$

Now, $1 + \frac{1}{x^2} = t^2$

$\therefore \frac{-2}{x^3}\,dx = 2t \cdot dt$

$\therefore \frac{dx}{x^3} = -t\,dt$

$I = \int t \cdot \log(t^2)\,(-t\,dt)$

$= \int -2t^2 \cdot \log t\,dt$

$I = -2 \int t^2 \cdot \log t\,dt$

$I = -2\,I_1 \qquad \text{... (1)}$

Now, $I_1 = \int t^2 \cdot \log t\,dt$

$u = \log t \quad ; \quad v = t^2$

$\therefore \frac{du}{dt} = \frac{1}{t}$

$I_1 = \log t \int t^2\,dt - \int \left[\frac{1}{t} \int t^2\,dt\right] dt$

$= \frac{\log t \cdot t^3}{3} - \int \frac{1}{t} \cdot \frac{t^3}{3}\,dt$

$= \frac{\log t \cdot t^3}{3} - \frac{1}{3}\int t^2\,dt$

$I_1 = \frac{\log t \cdot t^3}{3} - \frac{t^3}{9} + c$

Now, $1 + \frac{1}{x^2} = t^2$,

$\therefore t = \sqrt{1+\frac{1}{x^2}}$ yLku $t^3 = \left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}$

$I_1 = \frac{t^3}{3}\left[\log t - \frac{1}{3}\right] + c$

$I_1 = \frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}} \left[\log\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}} - \frac{1}{3}\right] + c_1$

$I = \frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}} \left[\frac{1}{2}\log\left[1+\frac{1}{x^2}\right] - \frac{1}{3}\right] + c_1$

Substitute the value of I_1 in equation (1),

$I = \frac{-2}{3}\left[1+\frac{1}{x^2}\right]^{\frac{3}{2}} \left[\frac{1}{2}\log\left(1+\frac{1}{x^2}\right) - \frac{1}{3}\right] + c$

$I = \frac{-1}{3}\left[1+\frac{1}{x^2}\right]^{\frac{3}{2}} \left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + c$

27.

Method 1 :

The given differential equation can be written as :

$$\left[xy\sin\left(\frac{y}{x}\right)-x^2\cos\left(\frac{y}{x}\right)\right]dy$$

$$=\left[xy\cos\left(\frac{y}{x}\right)+y^2\sin\left(\frac{y}{x}\right)\right]dx$$

$$\therefore\quad \frac{dy}{dx}=\frac{xy\cos\left(\frac{y}{x}\right)+y^2\sin\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right)-x^2\cos\left(\frac{y}{x}\right)}$$

Dividing numerator and denominator on RHS by x^2, we get,

$$\therefore\quad \frac{dy}{dx}=\frac{\frac{y}{x}\cos\left(\frac{y}{x}\right)+\left(\frac{y^2}{x^2}\right)\sin\left(\frac{y}{x}\right)}{\frac{y}{x}\sin\left(\frac{y}{x}\right)-\cos\left(\frac{y}{x}\right)} \quad \text{... (1)}$$

Clearly, equation (1) is a homogeneous differential equation of the form $\frac{dy}{dx}=g\left(\frac{y}{x}\right)$.

→ To solve it, we make the substitution

$$y = vx \quad \text{... (2)}$$

$$\therefore\quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\therefore\quad v+x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}$$

(using (1) and (2))

$$\therefore\quad x\frac{dv}{dx}=\frac{2v\cos v}{v\sin v-\cos v}$$

$$\therefore\quad \left(\frac{v\sin v-\cos v}{v\cos v}\right)dv=\frac{2dx}{x}$$

$$\int\left(\frac{v\sin v-\cos v}{v\cos v}\right)dv=2\int\frac{1}{x}dx$$

$$\therefore\quad \int\tan v\,dv-\int\frac{1}{v}dv=2\int\frac{1}{x}dx$$

$$\therefore\quad \log|\sec v|-\log|v|=2\log|x|+\log|c_1|$$

$$\therefore\quad \log\left|\frac{\sec v}{vx^2}\right|=\log|c_1|$$

$$\therefore\quad \frac{\sec v}{vx^2}=\pm c_1$$

→ Replacing v by $\frac{y}{x}$ in equation (3), we get,

$$\therefore\quad \frac{\sec\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)(x^2)}=c \qquad \text{where, } c=\pm c_1$$

$$\therefore\quad \sec\left(\frac{y}{x}\right)=cxy$$

which is the general solution of the given differential equation.

Method 2 :

$$\left(\frac{x\,dy-y\,dx}{x^2}\right)y\sin\frac{y}{x}=\left(\frac{y\,dx+x\,dy}{x}\right)\cos\frac{y}{x}$$

$$\therefore\quad d\left(\frac{y}{x}\right)\sin\frac{y}{x}=\frac{d(xy)}{xy}\cos\frac{y}{x}$$

$$\therefore\quad d\left(\frac{y}{x}\right)\tan\frac{y}{x}=\frac{d(xy)}{xy}$$

$$\therefore\quad \log\left|\sec\frac{y}{x}\right|=\log|cxy|$$

$$\therefore\quad \sec\frac{y}{x}=cxy$$